

MILNE

Irrational Number System

Mathematics and Astronomy

M. S.

1900

Learning and Labor.

LIBRARY

OF THE

University of Illinois.

CLASS.

BOOK.

VOLUME.

1900 M6

Accession No.





2853  
190

# THE IRRATIONAL NUMBER SYSTEM

By E. L. MILNE

Thesis for the Degree of MASTER OF SCIENCE in  
Mathematics and Astronomy

IN THE

UNIVERSITY OF ILLINOIS

PRESENTED MAY 31, 1900



Digitized by the Internet Archive  
in 2013

<http://archive.org/details/irrationalnumber00miln>

UNIVERSITY OF ILLINOIS

May 31, 1900

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Edward L. Milne

ENTITLED

The Irrational Number System

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF

Master of Science in

Mathematics and Astronomy

HEAD OF DEPARTMENT OF

Mathematics

S. H. Shattuck  
Prof. of Maths





A discussion of the system of rational numbers, including zero, shows that the four fundamental operations - addition, subtraction, multiplication and division, always lead to numbers of the system and to no new numbers. Hence if the rational number system were subjected to no operations but these four, it would be a system that would be complete in itself.

But, as is easily shown, if a number of the system is subjected to the operation inverse to raising to a power i.e., the extraction of a root, the resulting number may not be a number of the system. For example; the

equation 
$$x^2 = 2$$

can not have a solution in the rational number system.

For suppose it did have the solution  $x = \frac{a}{b}$

where  $a$  and  $b$  are relatively prime. Break  $a$  and  $b$  up into their prime factors;

$$x = \frac{a}{b} = \frac{p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_n}{q_1 \cdot q_2 \cdot q_3 \cdot \dots \cdot q_m}$$

$$\therefore p_1^2 \cdot p_2^2 \cdot p_3^2 \cdot \dots \cdot p_n^2 = 2 \cdot q_1^2 \cdot q_2^2 \cdot q_3^2 \cdot \dots \cdot q_m^2$$

and if we call this product  $M$  we have broken  $M$  up into its prime factors in two different ways, which is contrary to the fundamental theorem of number theory:

"A number can be broken up into its prime factors in only

Faint, illegible text, likely bleed-through from the reverse side of the page. The text is arranged in several paragraphs and is mostly unreadable due to low contrast and blurriness.

one way."

Again, if to the four elementary operations, the operation of limits is added, the resulting number may or may not be a number of the system. As an example; "If a systematic fraction is recurrent it has a rational limit; if it is not recurrent it does not have a rational limit."

("Allgemeine Arithmetik," by Dr. Otto Stolz, p.63, 66, 101)

A definition of the systematic fraction and the proof of this theorem will show that the rational number system is incomplete. If  $l$  is any positive integer,  $C_0$  any integer  $\geq 1$ ,  $C_1, C_2, C_3 \dots$  etc integers, the greatest of which is  $C_m$  where  $0 \leq C_m \leq l-1$ , and if

$$A = C_0 + \frac{C_1}{l} + \frac{C_2}{l^2} + \frac{C_3}{l^3} + \dots + \frac{C_m}{l^n} + \dots$$

$A$  is a systematic fraction. If the numerators after a certain term repeat in groups of  $K$ ,  $A$  is a recurrent systematic fraction.

Theorem;

If there exists a recurrent systematic fraction a rational number  $A$  can be found such that

$$S_n < A < S_n + \frac{1}{l^n}$$

I.E., the systematic fraction has a rational limit.



Let 
$$S_n = c_0 + \frac{c_1}{e} + \frac{c_2}{e^2} + \frac{c_3}{e^3} + \dots + \frac{c_m}{e^m}$$

and let 
$$S_m = c_0 + \frac{c_1}{e} + \frac{c_2}{e^2} + \dots + \frac{c_m}{e^m} \equiv \frac{Q}{e^m}$$

be the part that does not repeat. Let  $h$  be the period.

Let 
$$P = c_{m+1}e^{h-1} + c_{m+2}e^{h-2} + \dots + c_{m+h}$$

Form arbitrarily

$$A = \frac{(Qe^h + P) - Q}{(e^h - 1)e^m}$$

Then if  $h \neq 1$  and if  $P \neq e - 1$

$$S_n < A < S_n + \frac{1}{e^n}$$

If  $h = 1$  and  $P = e - 1$

$$(m) \quad S_n < \frac{Q+1}{e^m} \leq S_n + \frac{1}{e^n}$$

using upper sign for  $n < m$ . If period begins with  $c_1$ ,  $m = 0$ ,

$Q = c_0$   $\therefore$  equation (m) becomes

$$\frac{c_0 + 1}{e^0} = S_n + \frac{1}{e^n}$$

$$S_{n+k} \geq S_n \quad S_{n+k} + \frac{1}{e^{n+k}} \leq S_n + \frac{1}{e^n}$$

(STOLZ P.#64)

$$\begin{aligned} S_{m+rh} &= S_m + \frac{P}{e^{m+h}} + \frac{P}{e^{m+2h}} + \dots \\ &= S_m + \frac{P}{e^{m+h}} \cdot \frac{1-\omega^2}{1-\omega} \quad \left(\omega = \frac{1}{e^h}\right) \\ &= \frac{Q}{e^m} + \frac{P}{e^{m+h}} \cdot \frac{1-\omega^2}{1-\omega} \end{aligned}$$



$$S_{m+rh} = \frac{Qe^h - \omega Qe^h + P(1-\omega^r)}{e^{m+h}(1-\omega)}$$

$$= \frac{Qe^h - Q + P(1 - \frac{1}{e^{rh}})}{e^m(e^h - 1)}$$

We had

$$A = \frac{(Qe^h + P) - Q}{e^m(e^h - 1)}$$

$$\therefore S_{m+rh} = A - \frac{P}{e^{m+rh}(e^h - 1)}$$

or  $S_{m+rh} < A$  and no matter how great  $n$  is chosen,  $r$  can be chosen such that  $n < m+rh$

$$\therefore S_n \leq S_{m+rh} < A \quad \therefore S_n < A$$

Further (K)  $S_{m+rh} + \frac{1}{e^{m+rh}} = A + \frac{1}{e^{m+rh}} \left(1 - \frac{P}{e^h - 1}\right)$

if  $h \neq 1$  and  $P \neq e-1$  then  $P < e^h - 1$  for

$$P < (e-1) \{1 + e + e^2 + \dots + e^{h-1}\} = e^h - 1.$$

If

$$S_{m+rh} + \frac{1}{e^{m+rh}} > A$$

and  $\therefore$  if  $n \geq m+rh$

$$S_n + \frac{1}{e^n} > A$$

$$\therefore S_n < A < S_n + \frac{1}{e^n} \quad \text{if } h \neq 1 \text{ and } P \neq e-1$$

If  $h=1$  and  $P=e-1$

$$A = \frac{Q+1}{e^m} = \frac{S_m+1}{e^m} \quad \text{and by (K)}$$

$$S_{m+r} + \frac{1}{e^{m+r}} = A$$

If  $m \geq 1$   $e_m \neq e-1$

$$\therefore A = S_{m+r} + \frac{1}{e^{m+r}} < S_m + \frac{1}{e^m} \quad \text{if } n < m+r$$

Q. E. D.

1871

THE ... OF ...

...

...

...

...

...

...

...



Theorem;

If  $S_n$  has a rational limit it is a recurrent systematic fraction.

Hypothesis;

To every  $\varepsilon > 0$  belongs a  $\mu > 0$  such that

$$|\alpha - S_n| < \varepsilon \quad \text{if} \quad n > \mu$$

By definition of  $S_n$

$$0 \leq S_{n+r} - S_n \leq \frac{1}{e^n} - \frac{1}{e^{n+r}} < \frac{1}{e^n}$$

$$S_{n+r} \geq S_n \quad \text{and} \quad S_{n+r} + \frac{1}{e^{n+r}} \leq S_n + \frac{1}{e^n}$$

Fix  $n$ , call it  $m$ .

$S_{m+r}$  can be made greater than  $S_m$  by making  $r$  great enough.

$$\therefore S_{m+r} > S_m$$

$$\therefore S_{m+r+s} - \alpha > S_{m+r} - \alpha > S_m - \alpha$$

$$\therefore S_m \neq \alpha \quad \text{otherwise}$$

$$|S_{m+r+s} - \alpha| > \varepsilon \equiv S_{m+r} - \alpha$$

$$\text{i.e.} \quad |S_n - \alpha| > \varepsilon \quad \text{if} \quad n > m+r+s$$

which is contrary to hypothesis. It is easily seen intuitively that  $S_n < \alpha$ ; but to be rigid in the discussion it has to be proven.

In addition to the above inequalities,

$$\alpha - S_{m+r+s} > \alpha - \left( S_{m+r+s} + \frac{1}{e^{m+r+s}} \right) >$$

$$\alpha - \left( S_{m+r} + \frac{1}{e^{m+r}} \right) > \alpha - \left( S_m + \frac{1}{e^m} \right)$$

$$\text{from which} \quad S_m + \frac{1}{e^m} > \alpha$$

1917

It is a pleasure to have you here

and to see you all

again

and to see you all

is

to see you all

is

to see you all

and to see you all

and to see you all

is

and to see you all

is

is

and to see you all

and to see you all

and to see you all

and to see you all

1917

$$\therefore S_m < \alpha < S_m + \frac{1}{e^m}$$

But if this relation exists  $S_m$  is recurrent.

(Stolz P.#64)

We can also show the incompleteness of the rational number system by forming a number that does not belong to the system. For example; suppose the rational numbers have been arranged in some definite order and numbered  $\bar{c}_d$  (Crelle's "Journal für die reine und angewandte Mathematik." Vol.77, 1874. P.258 G. Cantor's Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen.")

Cantor defines a "Wertmenge" or "Punktmenge" as "abzählbar (enumerable) if we can assign them in a one to one correspondence to the positive integers. We will now show that the positive rational numbers can be counted. The positive rational numbers consist of the positive integers and the fractions of the form  $\frac{a}{b}$ , where  $a$  and  $b$  are relatively prime.

Now if we let  $|N| = a + b$ , give to  $N$  all possible positive integral values, choose the arrangements of the integers  $a$  and  $b$  as below, and form the corresponding rational numbers  $\frac{a}{b}$ , we will have these rational numbers in a definite order.

and it has been found that...

(1900-1901)

We have also seen the...

at which point...

the...

...

...

...

...

(1900-1901)

...

...

...

...

...

...

...

...

...

...

...

...

...

$N$	$a$	$b$	$\frac{a}{b}$
1	0, 1	1, 0	0
2	0, 1, 2	2, 1, 0	0, $\frac{1}{2}$
3	0, 1, 2, 3	3, 2, 1, 0	0, $\frac{1}{3}$ , $\frac{2}{3}$
4	0, 1, 2, 3, 4	4, 3, 2, 1, 0	0, $\frac{1}{4}$ , $\frac{2}{4}$ , $\frac{3}{4}$
5	0, 1, 2, 3, 4, 5	5, 4, 3, 2, 1, 0	0, $\frac{1}{5}$ , $\frac{2}{5}$ , $\frac{3}{5}$ , $\frac{4}{5}$
6	0, 1, 2, 3, 4, 5, 6	6, 5, 4, 3, 2, 1, 0	0, $\frac{1}{6}$ , $\frac{2}{6}$ , $\frac{3}{6}$ , $\frac{4}{6}$ , $\frac{5}{6}$
:	etc,	etc	etc
:	etc,	etc,	etc

We see that by taking  $N$  large enough we can include any rational number. Now taking the rational numbers in the order which they appear in the last columns of this table, and omitting any which appear and which have already been counted;

- The 1st is 0      0.000,000,000, - - - - -
  - The 2nd is 1      1.000,000,000, - - - - -
  - The 3rd is  $\frac{1}{2}$       0.500,000,000, - - - - -
  - The 4th is 2      2.000,000,000, - - - - -
  - The 5th is  $\frac{1}{3}$       0.333,333,333 - - - - -
  - The 6th is 3      3.000,000,000, - - - - -
  - The 7th is  $\frac{1}{4}$       0.250,000,000, - - - - -
  - The 8th is  $\frac{2}{3}$       0.666,666,666 - - - - -
  - The 9th is  $\frac{3}{2}$       1,500,000,000, - - - - -
- etc.etc.

Now if we form a number by always adding unity to the  $n$ th decimal place of the  $n$ th rational number, i.e.

.111,141,171, - - - - -

The first part of the report is devoted to a general survey of the  
 various methods of determining the relative humidity of air  
 and the various methods of determining the relative humidity of air  
 and the various methods of determining the relative humidity of air

Method	Accuracy	Cost
Psychrometer	± 0.1%	Low
Wet-bulb thermometer	± 0.2%	Low
Aspirator	± 0.3%	Low
Hygrometer	± 0.4%	Low
Thermohygrometer	± 0.5%	Low
Thermohygrometer	± 0.6%	Low
Thermohygrometer	± 0.7%	Low
Thermohygrometer	± 0.8%	Low
Thermohygrometer	± 0.9%	Low
Thermohygrometer	± 1.0%	Low

The second part of the report is devoted to a detailed description of the  
 various methods of determining the relative humidity of air and the  
 various methods of determining the relative humidity of air and the  
 various methods of determining the relative humidity of air

we have a number which evidently does not belong to the rational system.

Definition of  $\phi_n$

If a series of rational terms exists according to some law the result of taking the first  $n$  of these terms according to that law is represented by  $\phi_n$ .

Definition of Rational Limit of  $\phi_n$

Given a system of rational numbers  $\phi_1, \phi_2, \dots$  such that  $\phi_n$  depends upon the rational integer  $n$  for its value; and given that  $\alpha$  is a rational number. Then if to each positive rational number  $\epsilon > 0$  belongs a positive rational number  $\mu > 0$  such that

$$(a) \quad |\alpha - \phi_n| < \epsilon \quad \text{whenever } n > \mu \quad \text{i.e.}$$

$|\alpha - \phi_n| < \epsilon$  if  $n > \mu$ ,  $\alpha$  is called the limit of  $\phi_n$  as  $n$  increases indefinitely. Or as is usually abbreviated;

$$\alpha = \lim_{n \rightarrow \infty} \phi_n$$

Now given a series of rational numbers  $\phi_0, \phi_1, \phi_2, \dots$

$\phi_n$  depending upon the integer  $n$  for its value, it is required to determine whether the  $\phi$ 's have a rational limit or not. A necessary condition is at once derived from (a). (a) may be written

It is a common error to think that the  
the world is a flat surface.  
The earth is a sphere.

It is a common error to think that the  
the world is a flat surface.  
The earth is a sphere.

It is a common error to think that the  
the world is a flat surface.  
The earth is a sphere.

It is a common error to think that the  
the world is a flat surface.  
The earth is a sphere.

It is a common error to think that the  
the world is a flat surface.  
The earth is a sphere.



$$|\alpha - \varphi_{n+r}| < \varepsilon \quad \text{if } n > \mu \quad (r=1, 2, 3, \dots)$$

$$\therefore |\varphi_{n+r} - \varphi_n| < 2\varepsilon \quad \text{if } n > \mu \quad (r=1, 2, 3, \dots)$$

or since  $\varepsilon$  is arbitrary call  $2\varepsilon, \varepsilon$  and get

$$(b) |\varphi_{n+r} - \varphi_n| < \varepsilon \quad \text{if } n > \mu \quad (r=1, 2, 3, \dots)$$

as a necessary condition that the  $\varphi_n$ 's have a rational limit. But (b) is not a sufficient condition

as is seen from the fact that it is satisfied by both the recurrent and the non recurrent systematic fractions, and the latter do not have a rational limit.

Theorem:

If we have a set of  $\varphi_n$ 's such that having chosen an arbitrary rational number  $\varepsilon > 0$  we can determine a positive integer  $\mu$  such that

$$|\varphi_{n+r} - \varphi_n| < \varepsilon \quad \text{if } \begin{array}{l} n > \mu \\ r = 1, 2, 3, \dots \end{array}$$

one of three things will happen;

1. There will exist a

$$\text{such that } \begin{array}{l} \rho' > 0 \quad \text{and} \quad \nu > 0 \\ \varphi_n > \rho' \quad \text{if } n > \nu \quad \text{or} \end{array}$$

of these is whether will  
be a necessary condition for the  
existence of the other.  
It is clear that the  
existence of the one is  
not sufficient for the  
existence of the other.

It is clear that the  
existence of the one is  
not sufficient for the  
existence of the other.

It is clear that the  
existence of the one is  
not sufficient for the  
existence of the other.

2. There will exist a

$$-\rho < 0 \text{ and } \nu > 0$$

such that  $\phi_n < -\rho$  if  $n > \nu$

or we can choose

3.  $\epsilon > 0$  and find  $\nu > 0$  such that

$$-\epsilon < \phi_n < \epsilon \text{ if } n > \nu$$

Or for 3. we can say  $\phi_n$  approaches the limit 0.

In other words, under the given conditions, after enough  $\phi$ 's have been taken the remaining  $\phi$ 's are

(1) all positive and greater than a certain fixed quantity, or

(2) all negative and less than a certain fixed quantity, or

(3) they approach the limit zero.

Proof.  $|\phi_{n+r} - \phi_n| = \begin{cases} \phi_{n+r} - \phi_n \\ \phi_n - \phi_{n+r} \end{cases} < \epsilon \text{ if } n > \mu$   
 $r = 1, 2, 3, \dots$

(I)  $\therefore \phi_n - \epsilon < \phi_{n+r} < \phi_n + \epsilon \text{ if } n > \mu$   
 $r = 1, 2, 3, \dots$

Give  $\epsilon$  any positive value. Fix its value. Then

either  $\phi_n \leq \epsilon$  for all values of  $n > \mu$  or

there exists a particular  $n > \mu$  for which  $\phi_n > \epsilon$

2. There will be a...

and

is

the...

to be...

and...

the...

is

is

of the...

in other...

which...

(1) All...

of...

(2) All...

is...

(3) The...

There...

the...

for...

there...

If the latter is the case designate that particular value of  $n$  by  $m$  so  $\varphi_m > \varepsilon$  Then  $\varphi_m - \varepsilon = \rho'$

some positive number. But by (I)

$$\varphi_n - \varepsilon < \varphi_{n+r} \text{ if } n > \mu$$

hence  $\rho' < \varphi_{m+r}$  if  $m > \mu$

$$r = 1, 2, 3, \dots$$

and we have case 1.

But suppose there is no  $m$  for which  $\varphi_m > \varepsilon$ ; then we will always have  $\varphi_n \leq \varepsilon$  for all values of  $n > \mu$

On this supposition let  $\varepsilon' = \frac{\varepsilon}{2}$

Then by (I)

$$(I') \varphi_n - \varepsilon' < \varphi_{n+r} < \varphi_n + \varepsilon' \text{ if } n > \mu' > 0$$

$$r = 1, 2, 3, \dots$$

Again; either  $\varphi_n \leq \varepsilon'$  for all values of  $n > \mu'$  or there exists a particular  $n > \mu' > 0$  for which  $\varphi_n > \varepsilon'$

If this latter is the case, designate that particular value of  $n$  by  $m'$  so  $\varphi_{m'} > \varepsilon'$  Then  $\varphi_{m'} - \varepsilon' = \rho'$  some positive number. But by (I')

$$\varphi_n - \varepsilon' < \varphi_{n+r} \text{ if } n > \mu' > 0 \quad r = 1, 2, 3, \dots$$

hence  $\rho' < \varphi_{m'+r}$  if  $m' > \mu' > 0$

$$r = 1, 2, 3, \dots$$

and again we have case I. But if an  $m'$

can not be found such that  $\varphi_{m'} > \varepsilon'$  we will always have  $\varphi_n \leq \varepsilon'$  for all values of  $n > \mu'$

1. The first part of the document is a letter from the Secretary of the State to the President, dated 18th March 1847.

2. The second part is a report on the state of the country, dated 25th March 1847.

3.

4. The third part is a report on the state of the country, dated 25th March 1847.

5. The fourth part is a report on the state of the country, dated 25th March 1847.

6. The fifth part is a report on the state of the country, dated 25th March 1847.

7. The sixth part is a report on the state of the country, dated 25th March 1847.

8. The seventh part is a report on the state of the country, dated 25th March 1847.

9.

10. The eighth part is a report on the state of the country, dated 25th March 1847.

11. The ninth part is a report on the state of the country, dated 25th March 1847.

12. The tenth part is a report on the state of the country, dated 25th March 1847.

13. The eleventh part is a report on the state of the country, dated 25th March 1847.

14. The twelfth part is a report on the state of the country, dated 25th March 1847.

15.

16. The thirteenth part is a report on the state of the country, dated 25th March 1847.

17. The fourteenth part is a report on the state of the country, dated 25th March 1847.

18. The fifteenth part is a report on the state of the country, dated 25th March 1847.

19. The sixteenth part is a report on the state of the country, dated 25th March 1847.

Letting in turn  $\epsilon'' = \frac{\epsilon}{2^2}$ ,  $\epsilon''' = \frac{\epsilon}{2^3}$  etc.

and repeating the preceding reasoning, (notice that the  $\epsilon$ 's are always decreasing in value and approach zero as a limit) we get finally either

$$\varphi_n > \varphi_{m'} - \epsilon' \equiv \rho' \quad \text{if} \quad n \geq m' > \mu$$

or we will always have

$$\varphi_n \leq \epsilon' \quad \text{if} \quad n \geq m' > \mu \quad (\text{a})$$

Now starting with the right side of inequality (I)

and using a similar line of reasoning we get finally either

$$\varphi_n + \epsilon' < 0 \quad \text{if} \quad n \geq m' > \mu$$

and  $\therefore \varphi_n < -\epsilon' \equiv \rho'$  satisfying case 2, or  $\varphi_n + \epsilon'$

$\geq 0$  for all values of  $n \geq m' > \mu$

(b) i. e.  $\varphi_n \geq -\epsilon' \quad n \geq m' > \mu$

But if neither case 1 or case 2 is satisfied (a) and

(b) must be satisfied at the same time

(c)  $\therefore -\epsilon' \leq \varphi_n \leq \epsilon'$  for  $n \geq m' > \mu$

and  $\varphi_n$  approaches the limit zero.

Regular sequence.

$\varphi_n$  is called a regular sequence if having

chosen (1)  $\epsilon > 0$  we can find (2)  $\mu > 0$

such that  $|\varphi_{n+r} - \varphi_n| < \epsilon \quad r = 1, 2, 3, \dots$

whenever  $n > \mu$

The following is a list of the names of the members of the committee on the part of the University of Chicago.

1911

1912

1913

1914

The following is a list of the names of the members of the committee on the part of the University of Chicago.

1915

1916

1917

1918

1919

1920

The following is a list of the names of the members of the committee on the part of the University of Chicago.

1921

1922

1923

1924

1925

The following is a list of the names of the members of the committee on the part of the University of Chicago.

1926

1927

1928

1929



As an immediate consequence of the preceding theorem we have the following corollaries. (Proof in Stolz)

Corollary I. If a system of  $\varphi$ 's satisfy (I) Theorem 1 then  $\chi \pm \varphi_n$  and  $\chi \varphi_n$  satisfy (I) where  $\chi$  is any arbitrary rational number.

Cor. II.  $\frac{1}{\varphi_n}$  satisfies (I) except when  $\lim_{n \rightarrow \infty} \varphi_n = 0$  and then  $\lim_{n \rightarrow \infty} \frac{1}{\varphi_n} = \infty$

Cor. III. If  $\varphi_n$  and  $\psi_n$  both satisfy (I)  $\varphi_n \pm \psi_n$  and  $\psi_n \cdot \varphi_n$  will both satisfy (I); also  $\frac{\varphi_n}{\psi_n}$  satisfies (I) if  $\lim_{n \rightarrow \infty} \psi_n \neq 0$

Cor. IV. If  $\varphi_n, \psi_n, \dots, \omega_n$  all satisfy (I) then

$$\alpha \varphi_n \pm \beta \psi_n \pm \gamma \dots \pm \delta \omega_n \text{ and } K \cdot \varphi_n \cdot \psi_n \cdot \dots \cdot \omega_n \text{ satisfy (I)}$$

where  $K$  is an arbitrary rational number  $\neq 0$ , and  $\alpha, \beta, \dots$  are arbitrary rational numbers not all equal to zero at the same time.

Theorem:

If the  $\varphi$ 's have a rational limit  $\alpha$ , and if  $\chi$  is any rational quantity  $\neq 0$ , then  $\chi \varphi_n$  and  $\chi \pm \varphi_n$  have the rational limits  $\chi \cdot \alpha$  and  $\chi \pm \alpha$  respectively;

and if  $\alpha \neq 0$ ,  $\frac{1}{\varphi_n}$  has the rational limit  $\frac{1}{\alpha}$

Also if  $\psi_n$  has the rational limit  $\beta$ ,  $\psi_n \pm \varphi_n$

and  $\psi_n \cdot \varphi_n$  have the rational limits  $\beta \pm \alpha$

and  $\beta \cdot \alpha$  and if  $\lim_{n \rightarrow \infty} \psi_n = 0$   $\varphi_n : \psi_n$

as the principal component of the...  
we have the following...  
Theorem 1. Let  $f$  be a function...  
is an arbitrary...  
Theorem 2. Let  $f$  be a function...  
Theorem 3. Let  $f$  be a function...  
Theorem 4. Let  $f$  be a function...  
Theorem 5. Let  $f$  be a function...  
Theorem 6. Let  $f$  be a function...  
Theorem 7. Let  $f$  be a function...  
Theorem 8. Let  $f$  be a function...  
Theorem 9. Let  $f$  be a function...  
Theorem 10. Let  $f$  be a function...

has the rational limit,  $\alpha : \beta$

Corollary.

If a finite number of expressions  $\phi_n, \psi_n, etc$  each have a rational limit, the limit of the sum of these expressions is equal to the sum of their limits, and the limit of the product of these expressions is equal to the product of their limits.

Definition of the Irrational Number.

We will define the irrational number algebraically by saying that (I) Theorem 1 must be satisfied without the  $\phi$ 's having a rational limit.

If the rational numbers  $\phi_n$  satisfy (1) without having a rational limit we are led to a new object of thought - - a number different from each of these  $\phi$ 's. It is not expressible in terms of the  $\phi$ 's and nothing is stated about it further than that it satisfies (I) and is different from the rational  $\phi$ 's. Will represent this new object of thought by

$(\phi_n)$

and call it an Irrational number.

THE UNIVERSITY OF CHICAGO

CHICAGO, ILL.

DEPARTMENT OF CHEMISTRY

RECEIVED FROM THE UNIVERSITY OF CHICAGO  
LIBRARY ON APRIL 15, 1954  
BY THE UNIVERSITY OF CHICAGO  
LIBRARY ON APRIL 15, 1954

UNIVERSITY OF CHICAGO

LIBRARY OF THE UNIVERSITY OF CHICAGO  
540 EAST 57TH STREET  
CHICAGO, ILL. 60637

UNIVERSITY OF CHICAGO  
LIBRARY OF THE UNIVERSITY OF CHICAGO  
540 EAST 57TH STREET  
CHICAGO, ILL. 60637

UNIVERSITY OF CHICAGO  
LIBRARY OF THE UNIVERSITY OF CHICAGO  
540 EAST 57TH STREET  
CHICAGO, ILL. 60637

UNIVERSITY OF CHICAGO

Definition of Equality of Irrational  
Numbers.

$$(\varphi_n) = (\psi_n) \quad \text{if} \quad \lim_{n \rightarrow \infty} (\varphi_n - \psi_n) = 0$$

or if having chosen a rational number  $\varepsilon > 0$  we can find a rational integer  $\mu > 0$  such that  $|\varphi_n - \psi_n| < \varepsilon$  if  $n > \mu$

In order to put the subject on a rigid mathematical basis we will prove the following fundamental theorems regarding the equality of irrational numbers; - - which correspond to theorems for rational numbers ordinarily assumed in algebra and geometry to be axiomatic.

Theorem I.

$$\text{if} \quad (\varphi_n) = (\psi_n) \quad , \quad (\psi_n) = (\theta_n)$$

Theorem II.

$$\text{if} \quad (\varphi_n) = (\psi_n) \quad \text{and} \quad (\psi_n) = (\theta_n) \\ (\varphi_n) = (\theta_n)$$

Theorem III.

$$\text{if} \quad -\varepsilon < \varphi_n < \varepsilon \\ (\varphi_n) + (\psi_n) = (\theta_n)$$

Theorem IV.

$$\text{if the } \varphi_n \text{'s have a rational limit } \alpha, (\varphi_n) = \alpha$$

Theorem V.

If  $(\varphi_n)$  is an irrational number and if out of the sequence of  $\varphi_n$ 's we take a sequence  $\varphi_{k_1}, \varphi_{k_2}$ , etc,

$$(\varphi_{k_n}) = (\varphi_n)$$

DECLARATION OF INTEREST

STATE

11

and we have no other interest in the same.

IN WITNESS WHEREOF, we have hereunto set our hands and seals at the City of New York, this 11th day of June, 1911.

Witness my hand and seal this 11th day of June, 1911.

JOHN J. [Name], Mayor of the City and County of New York.

Attest: [Name], Deputy Mayor.

JOHN J. [Name], Mayor of the City and County of New York.

Attest: [Name], Deputy Mayor.

Chapter I.

1

Chapter II.

12

Chapter III.

13

Chapter IV.

14

Chapter V.

15

16

Proof The I.

Hypothesis is

$$|\varphi_n - \psi_n| < \varepsilon \text{ if } n > \mu$$

$$\therefore |\psi_n - \varphi_n| < \varepsilon \text{ if } n > \mu$$

and  $(\psi_n) = (\varphi_n)$  by the definition of equality of irrational numbers.

Theorem II.

$$\text{If } (\varphi_n) = (\psi_n) \quad \text{and } (\psi_n) = (\theta_n) \\ (\varphi_n) = (\theta_n)$$

$$\text{Hypothesis } |\varphi_n - \psi_n| < \varepsilon \text{ if } n > \mu$$

$$|\psi_n - \theta_n| < \varepsilon \text{ if } n > \mu'$$

$$\therefore |\varphi_n - \psi_n + \psi_n - \theta_n| < 2\varepsilon \text{ if } n > \bar{\mu}$$

$\bar{\mu}$  being either  $\mu$  or  $\mu'$ , whichever is the greater.

Now calling  $2\varepsilon, \kappa$

$$|\varphi_n - \theta_n| < \kappa \text{ if } n > \bar{\mu}$$

$$\therefore (\varphi_n) = (\theta_n) \text{ by definition.}$$

Theorem III

$$\text{If } -\varepsilon < \psi_n < \varepsilon \text{ if } n > \mu$$

$$(\varphi_n) + (\psi_n) = (\varphi_n)$$

$$|\varphi_n + \psi_n - \varphi_n| = |\psi_n| < \varepsilon \text{ if } n > \mu$$

$$\therefore (\varphi_n) + (\psi_n) = (\varphi_n) \text{ by definition.}$$

Page 10

Continued

1941

1942

1943

1944

1945

1946

1947

1948

1949

1950

1951

1952

1953

1954

1955

1956

1957

1958

1959

1960



Theorem IV.

If the  $Q$ 's have a rational limit  $\alpha$ ,  $(Q_n) = \alpha$

We evidently can not use our definition for equality of irrational numbers as there are none involved.

However, by the definition of a rational limit for the  $Q$ 's

$$|Q_n - \alpha| < \epsilon \text{ if } n > \mu$$

or  $\lim_{n \rightarrow \infty} Q_n = \alpha$

Stolz, if the  $Q$ 's have a rational limit to avoid complicating the notation, let's this limit also be represented by  $(Q_n)$

Theorem V.

If  $(Q_n)$  is an irrational number, and if out of the sequence of  $Q$ 's we take a sequence  $Q_{K_1}, Q_{K_2}, \text{ etc}$  these  $Q_{K_n}$ 's will form a sequence satisfying the same conditions as the  $Q$ 's did and  $(Q_n) = (Q_{K_n})$

$Q_{K_n}$  is a rational number, and we are to show that having chosen a positive rational number  $\epsilon$  at pleasure, we can find a positive rational integer  $\mu'$  such that

$$|Q_{K_{n'+r}} - Q_{K_{n'}}| < \epsilon \quad r = 1, 2, 3, \dots$$

for all values of  $n' > \mu'$

Since  $(Q_n)$  is irrational  $|Q_{n+r} - Q_n| < \epsilon$  for all values of  $n > \mu$

1900

... ..  
... ..  
... ..  
... ..

1901

1902

... ..  
... ..  
... ..

1903

... ..  
... ..  
... ..

... ..

... ..  
... ..  
... ..

1904

1905

... ..  
... ..  
... ..

Now choose  $\eta'$  so  $(\varphi_{K_{n'}}) > (\varphi_n) > (\varphi_\mu)$  as evidently can always be done. Call this value of  $n', \mu'$

Then  $|\varphi_{K_{\mu'+r}} - \varphi_{K_{\mu'}}| < \varepsilon$

and  $|\varphi_{K_{n'+r}} - \varphi_{K_{n'}}| < \varepsilon$  for all values of  $n' > \mu'$   
 $r = 1, 2, 3, \dots$  Q. E. D.

### Positive and Negative Irrational Numbers.

We define  $(\varphi_n) > \alpha$  ( $\alpha$  a rational number) if there exists a positive rational number  $\rho$  such that  $(\varphi_n - \alpha) > \rho$  if  $n$  is greater than some rational integer  $\mu > 0$ .

$\therefore (\varphi_n) > 0$  if there exists a positive rational number  $\rho$  such that  $(\varphi_n) > \rho$  if  $n > \mu$

Such an irrational number is called a positive irrational number.

We define  $(\varphi_n) < \alpha$  if there is a negative rational number  $-\rho$  such that  $(\varphi_n - \alpha) < -\rho$  for all values of  $n > \mu, \mu$  a positive integer.

$\therefore (\varphi_n) < 0$  if there exists a negative rational number  $-\rho$  such that  $(\varphi_n) < -\rho$  if  $n > \mu > 0$ ; such an irrational number is called a Negative Irrational Number.

$(\varphi_n) > (\psi_n)$  if there exists a positive rational number  $\rho$  such that  $(\varphi_n - \psi_n) > \rho$  for all values of  $n > \mu$

$(\varphi_n) < (\psi_n)$  if there exists a negative rational  $-\rho$

... ..

... ..

...

... ..

...

... ..

... ..

... ..

... ..

... ..

...

... ..

...

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

number such that  $\varphi_n - \psi_n < \rho$  for all values of  $n > \mu$ .  
 We will call the absolute value of  $(\varphi_n)$   $|(\varphi_n)| \therefore |(\varphi_n)|$   
 $-(\varphi_n)$  or  $(-\varphi_n)$  We will now prove the six fund-  
 amental theorems concerning the inequality of the  
 irrational numbers.

Theorem I.

If  $(\varphi_n) > (\psi_n)$   $(\psi_n) < (\varphi_n)$ .

Proof; If  $(\varphi_n) > (\psi_n)$ ,

$\varphi_n - \psi_n > \rho$  if  $n > \mu$

Then  $\psi_n - \varphi_n < -\rho$  if  $n > \mu$

and  $\therefore (\psi_n) < (\varphi_n)$

can prove similarly if  $(\varphi_n) \leq \alpha$

$\alpha \geq (\varphi_n)$

Theorem II.

Of two unequal irrational numbers, one  
 is greater and the other less.

Let  $(\varphi_n)$  and  $(\psi_n)$  be the unequal irrational numbers.

If  $|\varphi_n - \psi_n| < \varepsilon$  for all values of  $n > \mu$ ,  $(\varphi_n) = (\psi_n)$   
 by the definition of equality of irrational numbers.

But by hypothesis  $(\varphi_n) \neq (\psi_n)$

$\therefore |\varphi_n - \psi_n| \neq$  whenever  $n > \mu$

$\therefore |\varphi_n - \psi_n| \geq \varepsilon$  if  $n > \mu$

Now choose  $\rho < \varepsilon$ . Then either  $\varphi_n - \psi_n > \rho$  for all

January 1st, 1907.

My dear Mother

I will call the meeting at 10 o'clock on the 15th.

Love to all from your affectionate son,

John D. [Name]

Very truly yours,

[Signature]

John D. [Name]

15 [Address]

[City]

[State]

[Post Office]

Enclosed is

the amount of \$10.00 for the [purpose]

of the [organization]

for the [purpose]

of the [organization]

for the [purpose]

of the [organization]

values of  $n > \mu$  or  $\psi_n - \phi_n > \rho$  for all values of  $n > \mu$  and the theorem is proved by the definition of inequality of irrational numbers.

Theorem III.

$$\text{If } (\phi_n) > (\psi_n) \quad \text{and} \quad (\psi_n) = (\theta_n) \\ (\phi_n) > (\theta_n)$$

Proof  $\phi_n - \psi_n > \rho$  if  $n > \mu$

$$|\psi_n - \theta_n| < \varepsilon \quad \text{if } n > \mu'$$

Suppose  $|\psi_n - \theta_n| = \varepsilon' < \varepsilon$  if  $n > \mu'$

$$\therefore \psi_n = \theta_n \pm \varepsilon' \quad \text{if } n > \mu'$$

As  $\varepsilon$  is arbitrary we can make  $\varepsilon$  and  $\therefore \varepsilon' < \rho$

(a) Now  $\psi_n = \theta_n \pm \varepsilon' \quad \text{if } n > \mu' \quad \text{and}$

(b)  $\phi_n - \psi_n > \rho \quad \text{if } n > \mu$

(a) & (b) both hold if  $n > \bar{\mu}$ ,  $\bar{\mu}$  being whichever is the greater  $\mu$  or  $\mu'$ , or substituting in (b) for  $\psi_n$  its value from (a)

$$\phi_n - \theta_n > \rho \pm \varepsilon' \quad \text{if } n > \bar{\mu}$$

and  $(\phi_n) > (\theta_n)$  by definition.

Theorem IV.

$$\text{If } (\phi_n) > (\psi_n) \quad \text{and} \quad (\psi_n) > (\theta_n)$$

$$(\phi_n) > (\theta_n)$$

This theorem and theorem V can be proved in a manner exactly similar to that used in theorem III.

THE UNIVERSITY OF CHICAGO  
DEPARTMENT OF CHEMISTRY  
CHICAGO, ILLINOIS

January 11, 1954

Dear Mr. [Name]

I have received your letter of the 10th and am glad to hear that you are interested in the work of the Department of Chemistry at the University of Chicago. I am sure that you will find our work in the field of [Field] very interesting and I would like to discuss it with you if you are in the city.

Very truly yours,  
[Name]

Enclosure

Yours sincerely,  
[Name]

The University of Chicago  
Department of Chemistry  
57 South Dearborn Street  
Chicago 37, Illinois



Theorem V.

If  $(\varphi_n) > (\varphi'_n)$   
and  $(\psi_n) > (\psi'_n)$   
 $(\varphi_n) + (\psi_n) \not< (\varphi'_n) + (\psi_n)$  or  $(\varphi_n) + (\psi'_n)$

Theorem VI.

Between any two unequal irrational numbers there must exist an irrational number different from either of them.

Suppose  $(\varphi_n) > (\psi_n)$   
Then  $\varphi_n - \psi_n > \rho$  if  $n > \mu$   
or  $\varphi_n > \psi_n + \rho$  if  $n > \mu$   
and  $(\varphi_n) > (\psi_n) + \rho > (\psi_n)$

In a similar manner we can show, that between 0 and any irrational number there exists an irrational number. i.e. there is no smallest irrational number. Similarly there is no largest irrational number.

Corollary;

There is no smallest or largest rational number

Section 7.

32

and

30

Section 7.

There is no effect on the general principle of law  
which is not subject to the general principle of law

Article 10.

Section 7.

31

and

31

31

and

In a similar manner to the case, the law  
and the general principle of law  
which is not subject to the general principle of law  
which is not subject to the general principle of law

Section 7.

There is no effect on the general principle of law

and

### Addition of Irrational Numbers - - - - -

We will define  $(\ell_n) + \alpha$  or  $\alpha + (\ell_n)$  where  $(\ell_n)$  is irrational and  $\alpha$  rational, as the irrational number  $(\ell_n + \alpha)$ .

We will define  $(\ell_n) + (\psi_n)$  or  $(\psi_n) + (\ell_n)$ , where both  $(\ell_n)$  and  $(\psi_n)$  are irrational, as the irrational number  $(\ell_n + \psi_n)$ . From this definition and the laws for the addition of rational numbers we get when  $\alpha$  &  $\beta$  are real numbers, rational or irrational,

1.  $\alpha + \beta = \beta + \alpha$
2.  $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$
3. If  $\alpha = \alpha'$   $\alpha + \beta = \alpha' + \beta$
4. If  $\alpha > \alpha'$   $\alpha + \beta > \alpha' + \beta$
5. If  $\beta > 0$   $\alpha + \beta > \alpha$

That is, the laws for the addition of real numbers are the same as the laws for the addition of rational numbers.

Hence the laws for the subtraction of real numbers are the same as the laws for the subtraction of rational numbers.

### Multiplication of Irrational Numbers

We will define  $\alpha \cdot (\ell_n)$  or  $(\ell_n) \alpha$  where  $\alpha$  is rational and different from zero and  $(\ell_n)$  is irrational, as the irrational number  $(\alpha \ell_n)$ . We will define

$$0 \cdot (\ell_n) \text{ or } (\ell_n) \cdot 0 \text{ as } 0$$

ADDITION OF REGIONAL COURTS

The first step in the process of adding regional courts is to determine the geographical areas to be covered. This is done by dividing the country into regions of approximately equal population and geographical size. The next step is to select the locations for the regional courts. These should be centrally located within each region and accessible to the population. The final step is to establish the jurisdiction of the regional courts. This should be done in a way that ensures that the regional courts are able to handle the majority of cases that arise within their respective regions.

It is important to note that the addition of regional courts is a complex process that requires careful planning and coordination. It is essential to ensure that the regional courts are able to handle the workload that will be assigned to them. This may require the recruitment and training of additional judges and court staff. It is also important to ensure that the regional courts are able to interact effectively with the national court system.

RECOMMENDATIONS FOR THE ADDITION OF REGIONAL COURTS

Based on the findings of this study, the following recommendations are made for the addition of regional courts:

- 1. The geographical areas to be covered by the regional courts should be determined based on population and geographical size.
- 2. The locations for the regional courts should be centrally located within each region and accessible to the population.
- 3. The jurisdiction of the regional courts should be established in a way that ensures that they are able to handle the majority of cases that arise within their respective regions.

We will define  $(\varphi_n) \cdot (\psi_n)$  or  $(\psi_n) \cdot (\varphi_n)$ , where both are irrational, as the irrational number  $(\varphi_n \cdot \psi_n)$

From this definition and from the laws for the multiplication of rational numbers we get, if  $a$  and  $b$  are two real numbers,

$$(1) \quad a b = b a$$

$$(2) \quad (a b) c = a (b c)$$

$$(3) \quad (b + c) a = b a + c a$$

$$(4) \quad \text{If } a = a' \quad a b = a' b$$

$$(5) \quad \text{If } a > a' \quad \text{and } b > 0 \quad a b > a' b$$

That is the laws for the multiplication of real numbers are the same as the laws for the multiplication of rational numbers. Hence the fundamental laws for the division of real numbers are the same as the fundamental laws for the division of rational numbers

In order to show that the new system is complete we now have <sup>to show</sup> that the remaining operation of limits leads us to a number of the system.

Theorem;

A regular sequence of  $Q$ 's always has a limit in the new number system.

Hypothesis -  $|\varphi_{n+r} - \varphi_n| < \epsilon$  whenever  $n > \mu$   
 $r = 1, 2, 3, \dots$

... ..  
... ..  
... ..  
... ..

- (1)
- (2)
- (3)
- (4)
- (5)

... ..  
... ..  
... ..  
... ..  
... ..  
... ..  
... ..

... ..  
... ..

... ..  
... ..  
... ..

To prove; there exists a number  $A$  of the system such that having chosen (1)  $\varepsilon > 0$  we can determine (2)  $\mu > 0$  such that

$$|A - Q_n| < \varepsilon \text{ whenever } n > \mu$$

First; Let  $Q_n$  be a sequence of rational numbers.

$$|Q_{n+r} - Q_n| < \varepsilon \text{ for all values of } n > \sqrt{\quad}$$

$$r = 1, 2, 3, \dots$$

$$\therefore Q_n - \varepsilon < Q_{n+r} < Q_n + \varepsilon \text{ whenever } n > \sqrt{\quad}$$

$$r = 1, 2, 3, \dots$$

Choose for  $n$  a fixed value,  $n > \sqrt{\quad}$  call it  $m$ .

$$Q_m - \varepsilon < Q_{m+r} < Q_m + \varepsilon$$

Let  $r$  be a variable.

$Q_{m+r}$  is a regular sequence of rational numbers and by definition of irrational numbers it has a limit  $(Q_{m+r})$

Call  $(Q_{m+r})$ ,  $A$  and we will now prove  $A$  is the limit of  $Q_n$ .

$$Q_{m+r} - \varepsilon < A < Q_{m+r} + \varepsilon \text{ whenever } m > \sqrt{\quad}$$

$$\therefore |A - Q_{m+r}| < \varepsilon \text{ (} r = 1, 2, 3, \dots \text{) Let } (m+r) = n$$

$$\therefore |A - Q_n| < \varepsilon \text{ whenever } n > \sqrt{\quad}$$

hence  $A$  is the limit of  $Q_n$ .

$\therefore$  If  $Q_n$  is a regular sequence of rational numbers it has either a rational limit or by definition of irrational numbers an irrational limit.

... ..  
... ..  
... ..

... ..  
... ..

... ..

... ..  
... ..  
... ..  
... ..  
... ..  
... ..

... ..  
... ..  
... ..  
... ..



We must also prove it has only one limit.

Suppose it had another limit  $A + B = A'$

Then by definition of a limit we can choose (1)  $\epsilon > 0$   
and find (2)  $V' > 0$  such that  $|A - \phi_n| < \epsilon$  whenever  $n > V'$

But if it has another limit choosing the same  $\epsilon > 0$   
we can find  $V'' > 0$  such that

$$|A' - \phi_n| < \epsilon \text{ whenever } n > V''$$

Now calling the larger of  $V'$  and  $V''$ ,  $\bar{V}$  we  
have  $|A - \phi_n| < \epsilon$

and  $|A' - \phi_n| < \epsilon$  simultaneously for all  
values of  $n > \bar{V}$

$$\text{Now } |(A - \phi_n) - (A' - \phi_n)| \leq |A - \phi_n| + |A' - \phi_n|$$

since the absolute value of a difference is less than,  
or at most equal to the sum of the absolute values  
of its terms.

$$\therefore |A - A'| \leq 2\epsilon$$

$$\therefore |B| \leq 2\epsilon \quad \text{and since there}$$

is no least number, rational or irrational,  
and the regular sequence of rational  $\phi$ 's has one and  
only one limit.

But the proof is not complete for, as we have enlarged  
our number system, it is possible to imagine a regular



sequence of irrational numbers, and, therefore, we must show also that a regular sequence of irrational numbers has a limit in the system.

In order to prove this we need the following;

Lemma;

There always exists a rational number differing from any irrational number by an irrational number ~~by an irrational number~~ less than any positive quantity we may choose.

Let  $(\phi_n)$  be the irrational number, call it  $A$

Then by definition,

$$|\phi_{n+r} - \phi_n| < \epsilon \quad \text{whenever } n > \mu$$

$$r = 1, 2, 3, \dots$$

or

$$\phi_n - \epsilon < \phi_{n+r} < \phi_n + \epsilon$$

$$n > \mu$$

$$r = 1, 2, 3, \dots$$

But by first part of the theorem  $\phi_{n+r}$  has the limit  $(\phi_n) = A$

$$\therefore \phi_n - \epsilon < A < \phi_n + \epsilon \quad \text{if } n > \mu$$

$$\text{or } |A - \phi_n| < \epsilon \quad \text{whenever } n > \mu$$

and we can find a rational number  $\phi_n$  differing from  $A$  by an irrational quantity less than any positive quantity we may choose.

Now let the given regular sequence of irrational numbers be  $f_1, f_2, f_3, f_4$ , etc.

Let  $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4 > \epsilon_5$  etc. be a set of  $\epsilon$ 's

response to individual needs, and, therefore, we  
must know also that a similar response is required  
in order to give this to each individual.

(1944)

There is also a certain amount of  
the form and structure, which is a traditional  
but to be traditional, we must have the  
quantity of the response.

Let us now consider the  
form of the response.

10

Let us first look at the  
form of the response.

11

Let us now look at the  
form of the response. It is  
the form of the response, which is  
the form of the response, which is  
the form of the response, which is

12

13

chosen at pleasure

Form an arbitrary sequence  $g_1, g_2, g_3, g_4$  etc.

such that  $|g_n| < \epsilon_n$  and  $f_n - g_n = \alpha_n$

where  $\alpha_n$  is a rational number. This last condition is an allowable one by the preceding lemma

Will now show that  $\alpha_n$  is a regular sequence i.e.

having chosen (1)  $\delta > 0$  we can find a  $\mu > 0$

such that  $|\alpha_{n+r} - \alpha_n| < \delta$  whenever  $n > \mu$   
 $r = 1, 2, 3, \dots$

By hypothesis  $|f_{n+r} - f_n| < \epsilon$   $n > \mu$   
 $r = 1, 2, 3, \dots$

$$\therefore |f_{n+r} - f_n| = |\alpha_{n+r} + g_{n+r} - \alpha_n - g_n| < \epsilon \quad n > \mu$$

$$\text{or } \epsilon > |(\alpha_{n+r} - \alpha_n) + (g_{n+r} - g_n)|$$
$$\geq |\alpha_{n+r} - \alpha_n| - |g_{n+r} - g_n|$$

since the absolute value of a sum is greater than or at least equal to the difference of the absolute values of its terms.

$$\therefore |\alpha_{n+r} - \alpha_n| \leq |g_{n+r} - g_n| + \epsilon$$
$$\leq |g_{n+r}| + |g_n| + \epsilon$$
$$< \epsilon_{n+r} + \epsilon_n + \epsilon \quad \left. \begin{array}{l} n > \mu \\ r = 1, 2, 3, \dots \end{array} \right\}$$

Now choose arbitrarily a  $\delta > 0$

(K) Let  $\epsilon_n < \frac{\delta}{2}$  for all values of  $n > \nu'$

$\epsilon_{n+r} < \frac{\delta}{2}$  " " " "  $n > \nu'$

There is a certain amount of...  
and that...

is a national...  
will not...  
and that...

of the...

at least...  
and that...

The...

Then  $\epsilon_n + \epsilon_{n+r} < \delta$  for  $n > \sqrt{r}$   
 $\sqrt{r}$  being whichever is the greater  $\sqrt{r'}$  or  $\sqrt{r''}$

Let  $\delta = \epsilon_{n+r} + \epsilon_n + \epsilon$

This equation fixes the value of  $\epsilon$  and by (K)

there exists a  $\mu > 0$  such that

$$|\phi_{n+r} - \phi_n| < \delta \quad \text{if } n > \mu$$

$r = 1, 2, 3, \dots$

$\therefore \phi_n$  is a regular sequence and has the limit  $(\phi_n) \equiv A$

Will now show that  $A$  is the limit of  $f_n$  or having

chosen (1)  $\delta > 0$  we can find (2)  $\mu > 0$  such

that  $|A - f_n| < \delta \quad n > \mu$

$$|A - \phi_n| < \epsilon \quad n > \lambda$$

$$|A - f_n + g_n| < \epsilon \quad n > \lambda$$

$$|A - f_n| - |g_n| \leq |A - f_n + g_n| < \epsilon \quad n > \lambda$$

$$|A - f_n| < \epsilon + |g_n| \quad \text{if } n > \lambda$$

$$< \epsilon + \epsilon_n < \delta \quad \text{if } n > \lambda$$

$\therefore A$  is the limit of  $f_n$ .

Q. E. D.

and the irrational number system is complete.







UNIVERSITY OF ILLINOIS-URBANA



3 0112 082194769