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Irrational Number System

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THE IRRATIONAL NUMBER SYSTEM

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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A discussion of the system of rational numbers, including zero, shows that the four fundamental operations - addition, subtraction, multiplication and division, always lead to numbers of the system and to no new numbers. Hence if the rational number system were subjected to no operations but these four, it would be a system that would be complete in itself.

But, as is easily shown, if a number of the system is subjected to the operation inverse to raising to a power i.e., the extraction of a root, the resulting number may not be a number of the system. For example; the equation

$$x^2 = 2$$

can not have a solution in the rational number system. For suppose it did have the solution $x = \frac{a}{b}$ where a and b are relatively prime. Break a and b up into their prime factors;

$$x = \frac{a}{b} = \frac{p_1 \cdot p_2 \cdot p_3 \cdot \cdots\cdots p_n}{q_1 \cdot q_2 \cdot q_3 \cdot \cdots\cdots q_m}$$

$$\therefore\ p_1^2 \cdot p_2^2 \cdot p_3^2 \cdot \cdots\cdots p_n^2 = 2\, q_1^2 \cdot q_2^2 \cdot q_3^2 \cdot \cdots q_m^2$$

and if we call this product M we have broken M up into its prime factors in two different ways, which is contrary to the fundamental theorem of number theory: "A number can be broken up into its prime factors in only

one way."

Again, if to the four elementary operations, the operation of limits is added, the resulting number may or may not be a number of the system. As an example; "If a systematic fraction is recurrent it has a rational limit; if it is not recurrent it does not have a rational limit."
("Allgemeine Arithmetik," by Dr. Otto Stolz, p.63, 66, 101)
A definition of the systematic fraction and the proof of this therom will show that the rational number system is incomplete. If ℓ is any positive integer, C_0 any integer $\gtrless 1$, c_1, c_2, c_3 --- etc integers, the greatest of which is C_m where $0 \leqq C_m \leqq \ell - 1$, and if

$$A = C_0 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \frac{c_3}{\ell^3} + \cdots + \frac{c_n}{\ell^n} + \cdots$$

A is a systematic fraction. If the numerators after a certain term repeat in groups of K, A is a recurrent systematic fraction.
Theorem;

If there exists a recurrent systematic fraction a rational number A can be found such that

$$S_n < A < S_n + \frac{1}{\ell^n}$$

I.E., the systematic fraction has a rational limit.

Let $$S_n = c_0 + \frac{c_1}{e} + \frac{c_2}{e^2} + \frac{c_3}{e^3} + \cdots + \frac{c_n}{e^n}$$

and let $$S_m = c_0 + \frac{c_1}{e} + \frac{c_2}{e^2} + \cdots + \frac{c_m}{e^m} \equiv \frac{Q}{e^m}$$

be the part that does not repeat. Let h be the period.

Let $$P = c_{m+1}e^{h-1} + c_{m+2}e^{h-2} + \cdots + c_{m+h}$$

Form arbitrarily
$$A = \frac{(Qe^h + P) - Q}{(e^h - 1)e^m}$$

Then if $h \neq 1$ and if $P \neq e-1$
$$S_n < A < S_n + \frac{1}{e^n}$$

If $h = 1$ and $P = e-1$

(m) $$S_n < \frac{Q+1}{e^m} \leqq S_n + \frac{1}{e^n}$$

using upper sign for $n < m$. If period begins with c_1 $m = 0$, $Q = c_0$ $\therefore$ equation (m) becomes
$$\frac{c_0+1}{e^0} = S_n + \frac{1}{e^n}$$

$$S_{n+k} \geqq S_n \qquad S_{n+k} + \frac{1}{e^{n+k}} \leqq S_n + \frac{1}{e^n}$$

(STOLZ P.#64)

$$S_{m+rh} = S_m + \frac{P}{e^{m+h}} + \frac{P}{e^{m+2h}} + \cdots$$
$$= S_m + \frac{P}{e^{m+h}} \cdot \frac{1-\omega^r}{1-\omega} \qquad \left(\omega = \frac{1}{e^h}\right)$$
$$= \frac{Q}{e^m} + \frac{P}{e^{m+h}} \cdot \frac{1-\omega^r}{1-\omega}$$

$$S_{m+rh} = \frac{Qe^h - \omega Qe^h + P(1-\omega^r)}{e^{m+h}(1-\omega)}$$

$$= \frac{Qe^h - Q + P\left(1 - \frac{1}{e^{rh}}\right)}{e^m(e^h-1)}$$

We had

$$A = \frac{(Qe^h+P) - Q}{e^m(e^h-1)}$$

$$\therefore S_{m+rh} = A - \frac{P}{e^{m+rh}(e^h-1)}$$

or $S_{m+rh} < A$ and no matter how great n is chosen, r can be chosen such that $n < m+rh$

$$\therefore S_n \leqq S_{m+rh} < A \qquad \therefore S_n < A$$

Further

$$(K) \quad S_{m+rh} + \frac{1}{e^{m+rh}} = A + \frac{1}{e^{m+rh}}\left(1 - \frac{P}{e^h-1}\right)$$

if $h \neq 1$ and $P \neq e-1$ then $P < e^h-1$ for

$$P < (e-1)\{1+e+e^2+ \cdots + e^{h-1}\} = e^h-1.$$

If $P < e^h-1$

$$S_{m+rh} + \frac{1}{e^{m+rh}} > A$$

and $\therefore$ if $n \geqq m+rh$

$$S_n + \frac{1}{e^n} > A$$

$$\therefore \quad S_n < A < S_n + \frac{1}{e^n} \quad \text{if } h \neq 1 + P \neq e-1$$

If $h=1$ and $P = e-1$

$$A = \frac{Q+1}{e^m} = \frac{S_m+1}{e^m} \quad \text{and by } (K)$$

$$S_{m+r} + \frac{1}{e^{m+r}} = A$$

If $m \geqq 1$ $\quad c_m \neq e-1$

$$\therefore A = S_{m+r} + \frac{1}{e^{m+r}} < S_n + \frac{1}{e^n} \quad \text{if } n < m+r$$

Q. E. D.

Theorem;

If S_n has a rational limit it is a recurrent systematic fraction.

Hypothesis;

To every $\varepsilon > 0$ belongs a $\mu > 0$ such that

$|\alpha - S_n| < \varepsilon$ if $n > \mu$

By definition of S_n

$$0 \leqq S_{n+r} - S_n \leqq \frac{1}{e^n} - \frac{1}{e^{n+r}} < \frac{1}{e^n}$$

$$S_{n+r} \geqq S_n \quad \text{and} \quad S_{n+r} + \frac{1}{e^{n+r}} \leqq S_n + \frac{1}{e^n}$$

Fix n, call it m.

S_{m+r} can be made greater than S_m by making r great enough.

$$\therefore S_{m+r} > S_m$$

$$\therefore S_{m+r+s} - \alpha > S_{m+r} - \alpha > S_m - \alpha$$

$$\therefore S_m \not\geqq \alpha \qquad \text{otherwise}$$

$$|S_{m+r+s} - \alpha| > \varepsilon \equiv S_{m+r} - \alpha$$

i.e. $|S_n - \alpha| > \varepsilon$ if $n > m + r + s$

which is contrary to hypothesis. It is easily seen intuitionally that $S_n < \alpha$; but to be rigid in the discussion it has to be proven.

In addition to the above inequalities,

$$\alpha - S_{m+r+s} > \alpha - \left(S_{m+r+s} + \frac{1}{e^{m+r+s}}\right) >$$

$$\alpha - \left(S_{m+r} + \frac{1}{e^{m+r}}\right) > \alpha - \left(S_m + \frac{1}{e^m}\right)$$

from which $S_m + \frac{1}{e^m} > \alpha$

$$\therefore S_m < \alpha < S_m + \frac{1}{e^m}$$

But if this relation exists S_m is recurrent.

(Stolz P.#64)

We can also show the incompleteness of the tational number system by forming a number that does not belong to the system. For example; suppose the rational numbers have been arranged in some definite order and numbered (Crelle's "Journal für die reine und angewandte Mathematik." Vol.77, 1874. P.258 G. Cantor's Ueber eine Eigenschaft des Inbegriffes aller reelen algebraischen Zahlen.")

Cantor defines a "Wertmenge" or "Punktmenge" as "abzahlbar (enumerable) if we can assign them in a one to one correspondence to the positive integers. We will now show that the positive rational numbers can be counted. The positive rational numbers consist of the positive integers and the fractions of the form $\frac{a}{b}$, where a and b are relatively prime.

Now if we let $|N| = a + b$, give to N all possible positive integral values, choose the arrangements of the integers a and b as below, and form the corresponding rational numbers $\frac{a}{b}$, we will have these rational numbers in a definite order.

N	a	b	$\frac{a}{b}$
1	0, 1	1, 0	0
2	0, 1, 2	2, 1, 0	0, 1
3	0, 1, 2, 3	3, 2, 1, 0	0, 1/2, 2
4	0, 1, 2, 3, 4	4, 3, 2, 1, 0	0, 1/3, 1, 3
5	0, 1, 2, 3, 4, 5	5, 4, 3, 2, 1, 0	0, 1/4, 2/3, 3/2, 4
6	0, 1, 2, 3, 4, 5, 6	6, 5, 4, 3, 2, 1, 0	0, 1/5, 1/2, 1, 2, 5
⋮	etc,	etc	etc
⋮	etc,	etc.	etc

We see that by taking N large enough we can include any rational number. Now taking the rational numbers in the order which they appear in the last columns of this table, and omitting any which appear and which have already been counted;

The 1st is 0 0.000,000,000, - - - - - - - -

The 2nd is 1 1.000,000,000, - - - - - - - -

The 3rd is ½ 0.500,000,000, - - - - - - - -

The 4th is 2 2.000,000,000, - - - - - - - -

The 5th is 1/3 0.333,333,333 - - - - - - - - -

The 6th is 3 3.000,000,000,- - - - - - - - -

The 7th is ¼ 0.250,000,000,- - - - - - - - -

The 8th is 2/3 0.666,666,666 - - - - - - - - -

The 9th is 3/2 1,500,000,000, - - - - - - - - -

etc.etc.

Now if we form a number by always adding unity to the *nth* decimal place of the *nth* rational number, i.e.

.111,141,171, - - - - - - - - - - - -

we have a number which evidently does not belong to the rational system.

Definition of φ_n

If a series of rational terms exists according to some law the result of taking the first n of these terms according to that law is represented by φ_n.

Definition of Rational Limit of φ_n

Given a system of rational numbers φ_1, φ_2; etc such that φ_n depends upon the rational integer n for its value; and given that α is a rational number. Then if to each positive rational number $\varepsilon > 0$ belongs a positive rational number $\mu > 0$ such that

(a) $|\alpha - \varphi_n| < \varepsilon$ whenever $n > \mu$ i.e. $|\alpha - \varphi_n| < \varepsilon$ if $n > \mu$, α is called the limit of φ_n as n increases indefinitely. Or as is usually abbreviated;

$$\alpha = \lim_{n=\infty} \varphi_n$$

Now given a series of rational numbers $\varphi_0, \varphi_1, \varphi_2, ---$ φ_n depending upon the integer n for its value, it is required to determine whether the φ's have a rational limit or not. A necessary condition is at once derived from (a). (a) may be written

$$|\alpha - \varphi_{n+r}| < \varepsilon \quad \text{if } n > \mu \quad (r = 1, 2, 3, \cdots)$$

$$\therefore \; |\varphi_{n+r} - \varphi_n| < 2\varepsilon \quad \text{if } n > \mu \quad (r = 1, 2, 3, \cdots)$$

or since ε is arbitrary call 2ε, ε and get

$$(b) \quad |\varphi_{n+r} - \varphi_n| < \varepsilon \text{ if } n > \mu \quad (r = 1, 2, 3, \cdots)$$

as a necessary condition that the φ's have a rational limit. But (b) is not a sufficient condition as is seen from the fact that it is satisfied by both the recurrent and the non recurrent systematic fractions, and the latter do not have a rational limit.

Theorem:

If we have a set of φ's such that having chosen an arbitrary rational number $\varepsilon > 0$ we can determine a positive integer μ such that

$$|\varphi_{n+r} - \varphi_n| < \varepsilon \text{ if } \begin{array}{l} n > \mu \\ r = 1, 2, 3 \cdots \end{array}$$

one of three things will happen;

1. There will exist a

$$\rho' > 0 \quad \text{and} \quad \nu > 0$$

such that $\varphi_n > \rho'$ if $n > \nu$ or

2. There will exist a

$$-\rho < 0 \text{ and } \nu > 0$$

such that $\varphi_n < -\rho$ if $n > \nu$

or we can choose

3. $\varepsilon > 0$ and find $\nu > 0$ such that

$$-\varepsilon < \varphi_n < \varepsilon \quad \text{if} \quad n > \nu$$

Or for 3. we can say φ_n approaches the limit 0.

In other words, under the given conditions, after enough φ's have been taken the remaining φ's are

(1) all positive and greater than a certain fixed quantity, or

(2) all negative and less than a certain fixed quantity, or

(3) they approach the limit zero.

Proof. $$|\varphi_{n+r} - \varphi_n| = \begin{Bmatrix} \varphi_{n+r} - \varphi_n \\ \varphi_n - \varphi_{n+r} \end{Bmatrix} < \varepsilon \quad \text{if } n > \mu, \; r = 1, 2, 3, \cdots$$

$$(I) \quad \therefore \varphi_n - \varepsilon < \varphi_{n+r} < \varphi_n + \varepsilon \qquad n > \mu, \; r = 1, 2, 3, \cdots$$

Give ε any positive value. Fix its value. Then either $\varphi_n \leqq \varepsilon$ for all values of $n > \mu$ or there exists a particular $n > \mu$ for which $\varphi_n > \varepsilon$

If the latter is the case designate that particular value of n by m so $\varphi_m > \varepsilon$. Then $\varphi_m - \varepsilon = \rho'$ some positive number. But by (I)

$$\varphi_n - \varepsilon < \varphi_{n+r} \quad \text{if} \quad n > \mu$$

hence $\rho' < \varphi_{m+r}$ if $m > \mu$, $r = 1, 2, 3, ---$

and we have case 1.

But suppose there is no m for which $\varphi_m > \varepsilon$; then we will always have $\varphi_n \leqq \varepsilon$ for all values of $n > \mu$

On this supposition let $\varepsilon' = \frac{\varepsilon}{2}$

Then by (I)

$$(I') \quad \varphi_n - \varepsilon' < \varphi_{n+r} < \varphi_n + \varepsilon' \quad \text{if} \quad n > \mu' > 0, \quad r = 1, 2, 3, ---$$

Again; either $\varphi_n \leqq \varepsilon'$ for all values of $n > \mu'$ or there exists a particular $n > \mu' > 0$ for which $\varphi_n > \varepsilon'$

If this latter is the case, designate that particular value of n by m' so $\varphi_{m'} > \varepsilon'$ Then $\varphi_{m'} - \varepsilon' = \rho'$ some positive number. But by (I')

$$\varphi_n - \varepsilon' < \varphi_{n+r} \quad \text{if} \quad n > \mu' > 0 \quad r = 1, 2, 3, ---$$

hence $\rho' < \varphi_{m'+r}$ if $m' > \mu' > 0$, $r = 1, 2, 3, ---$

and again we have case I. But if an m' can not be found such that $\varphi_{m'} > \varepsilon'$ we will always have $\varphi_n \leqq \varepsilon'$ for all values of $n > \mu'$

Letting in turn $\varepsilon'' = \frac{\varepsilon}{2^2}$, $\varepsilon''' = \frac{\varepsilon}{2^3}$ etc.

and repeating the preceding reasoning, (notice that the ε's are always decreasing in value and approach zero as a limit) we get finally either

$$\varphi_n > \varphi_{m^v} - \varepsilon^v \equiv \rho' \quad \text{if} \quad n \geqq m^v > \mu$$

or we will always have

$$\varphi_n \leqq \varepsilon^v \quad \text{if} \quad n \geqq m^v > \mu \quad \text{(a)}$$

Now starting with the right side of inequality (I) and using a similar line of reasoning we get finally either

$$\varphi_n + \varepsilon^v < 0 \quad \text{if} \quad n \geqq m^v > \mu$$

and $\therefore \varphi_n < -\varepsilon^v \equiv \rho'$ satisfying case 2, or $\varphi_n + \varepsilon^v \geqq 0$ for all values of $n \geqq m^v > \mu$

(b) i. e. $\varphi_n \geqq -\varepsilon^v \quad n \geqq m^v > \mu$

But if neither case 1 or case 2 is satisfied (a) and (b) must be satisfied at the same time

(c) $\therefore -\varepsilon^v \leqq \varphi_n \leqq \varepsilon^v$ for $n \geqq m^v > \mu$

and φ_n approaches the limit zero.

Regular sequence.

φ_n is called a regular sequence if having chosen (1) $\varepsilon > 0$ we can find (2) $\mu > 0$ such that $|\varphi_{n+r} - \varphi_n| < \varepsilon \quad r = 1, 2, 3, \ldots$

whenever $n > \mu$

As an immediate consequence of the preceding theorem we have the following corollaries. (Proof in Stolz)

Corollary 1. If a system of φ's satisfy (I) Theorem 1 then $\gamma \pm \varphi_n$ and $\gamma \varphi_n$ satisfy (I) where γ is any arbitrary rational number.

Cor. 11. $\frac{1}{\varphi_n}$ satisfies (I) except when $\lim \varphi_n = 0_{n=\infty}$ and then $\lim_{n=\infty} \frac{1}{\varphi_n} = \infty$

Cor. 111. If φ_n and ψ_n both satisfy (I) $\varphi_n \pm \psi_n$ and $\psi_n \cdot \varphi_n$ will both satisfy (I); also $\frac{\varphi_n}{\psi_n}$ satisfies (I) if $\text{limit}_{n=\infty} \psi_n \neq 0$

Cor. 1V. If $\varphi_n, \psi_n --- \vartheta_n$ all satisfy (I) then

$\alpha \varphi_n \pm \beta \psi_n \pm \gamma \chi_n ---- \cdot \pm \delta \vartheta_n$ and

$K \cdot \varphi_n \cdot \psi_n \cdot \chi_n \cdot --- \cdot \vartheta_n$ satisfy (I)

where K is an arbitrary rational number $\neq 0$, and $\alpha, \beta,$ etc are arbitrary rational numbers not all equal to zero at the same time.

Theorem:

If the φ's have a rational limit α, and if γ is any rational quantity $\neq 0$, then $\gamma \cdot \varphi_n$ and $\gamma \pm \varphi_n$ have the rational limits $\gamma \cdot \alpha$ and $\gamma \pm \alpha$ respectively;

and if $\alpha \neq 0$, $\frac{1}{\varphi_n}$ has the rational limit $\frac{1}{\alpha}$

Also if ψ_n has the rational limit β, $\psi_n \pm \varphi_n$ and $\psi_n \cdot \varphi_n$ have the rational limits $\beta \pm \alpha$ and $\beta \cdot \alpha$ and if $\lim_{n=\infty} \psi_n = 0$ $\varphi_n : \psi_n$

has the rational limit, $\alpha : \beta$

Corollary.

If a finite number of expressions $\varphi_n, \psi_n,$ etc each have a rational limit, the limit of the sum of these expressions is equal to the sum of their limits, and the limit of the product of these expressions is equal to the product of their limits.

Definition of the Irrational Number.

We will define the irrational number algebraically by saying that (I) Theorem 1 must be satisfied without the φ's having a rational limit.

If the rational numbers φ_n satisfy (1) without having a rational limit we are led to a new object of thought - - a number different from each of these φ's. It is not expressible in terms of the φ's and nothing is stated about it further than that it satisfies (I) and is different from the rational φ's. Will represent this new object of thought by

$$(\varphi_n)$$

and call it an Irrational number.

Definition of Equality of Irrational Numbers.

$$(\varphi_n) = (\psi_n) \quad \text{if} \quad \lim_{n=\infty} (\varphi_n - \psi_n) = 0$$

or if having chosen a rational number $\varepsilon > 0$ we can find a rational integer $\mu > 0$ such that $|\varphi_n - \psi_n| < \varepsilon$ if $n > \mu$

In order to put the subject on a rigid mathematical basis we will prove the following fundamental theorems regarding the equality of irrational numbers; - - which correspond to theorems for rational numbers ordinarily assumed in algebra and geometry to be axiomatic.

Theorem 1.

If $(\varphi_n) = (\psi_n)$, $(\psi_n) = (\varphi_n)$

Theorem 11.

If $(\varphi_n) = (\psi_n)$ and $(\psi_n) = (\theta_n)$

$$(\varphi_n) = (\theta_n)$$

Theorem 111.

If $-\varepsilon < \psi_n < \varepsilon$

$$(\varphi_n) + (\psi_n) = (\varphi_n)$$

Theorem 1V.

If the φ's have a rational limit α, $(\varphi_n) = \alpha$

Theorem V.

If (φ_n) is an irrational number and if out of the sequence of φ's we take a sequence $\varphi_{k_1}, \varphi_{k_2}$ etc,

$$(\varphi_{k_n}) = (\varphi_n)$$

Proof The I.

Hypothesis is

$$|\varphi_n - \psi_n| < \varepsilon \text{ if } n > \mu$$

$$\therefore |\psi_n - \varphi_n| < \varepsilon \text{ if } n > \mu$$

and $(\psi_n) = (\varphi_n)$ by the definition of equality of irrational numbers.

Theorem II.

If $(\varphi_n) = (\psi_n)$ and $(\psi_n) = (\theta_n)$

$$(\varphi_n) = (\theta_n)$$

Hypothesis $|\varphi_n - \psi_n| < \varepsilon$ if $n > \mu$

$$|\psi_n - \theta_n| < \varepsilon \text{ if } n > \mu'$$

$$\therefore |\varphi_n - \psi_n + \psi_n - \theta_n| < 2\varepsilon \text{ if } n > \bar{\mu}$$

$\bar{\mu}$ being either μ or μ', whichever is the greater.

Now calling 2ε, κ

$$|\varphi_n - \theta_n| < \kappa \text{ if } n > \bar{\mu}$$

$$\therefore (\varphi_n) = (\theta_n) \text{ by definition.}$$

Theorem III

If $-\varepsilon < \psi_n < \varepsilon$ if $n > \mu$

$$(\varphi_n) + (\psi_n) = (\varphi_n)$$

$$|\varphi_n + \psi_n - \varphi_n| = |\psi_n| < \varepsilon \text{ if } n > \mu$$

$$\therefore (\varphi_n) + (\psi_n) = (\varphi_n) \text{ by definition.}$$

Theorem IV.

If the φ's have a rational limit $\alpha, (\varphi_n) = \alpha$

We evidently can not use our definition for equality of irrational numbers as there are none involved. However, by the definition of a rational limit for the φ's

$$|\varphi_n - \alpha| < \varepsilon \text{ if } n > \mu$$

or

$$\lim_{n=\infty} \varphi_n = \alpha$$

Stolz, if the φ's have a rational limit to avoid complicating the notation, lets this limit also be represented by (φ_n)

Theorem V.

If (φ_n) is an irrational number, and if out of the sequence of φ's we take a sequence $\varphi_{K_1}, \varphi_{K_2}$, etc these φ_{K_n}'s will form a sequence satisfying the same conditions as the φ's did and $(\varphi_n) = (\varphi_{K_n})$

φ_{K_n} is a rational number, and we are to show that having chosen a positive rational number ε at pleasure, we can find a positive rational integer μ' such that

$$|\varphi_{K_{n'+r}} - \varphi_{K_{n'}}| < \varepsilon \qquad r = 1, 2, 3, ---$$

for all values of $n' > \mu'$

Since (φ_n) is irrational $|\varphi_{n+r} - \varphi_n| < \varepsilon$ for all values of $n > \mu$

Now choose n' so $\varphi_{K_{n'}} > \varphi_n > \varphi_\mu$ as evidently can always be done. Call this value of n', μ'

Then $|\varphi_{K_{\mu'+r}} - \varphi_{K_{\mu'}}| < \varepsilon$

and $|\varphi_{K_{n'+r}} - \varphi_{K_{n'}}| < \varepsilon$ for all values of $n' > \mu'$

$r = 1, 2, 3, \ldots$ Q. E. D.

Positive and Negative Irrational Numbers.

We define $(\varphi_n) > \alpha$ (α a rational number) if there exists a positive rational number ρ such that $\varphi_n - \alpha > \rho$ if n is greater than some rational integer $\mu > 0$.

$\therefore (\varphi_n) > 0$ if there exists a positive rational number ρ such that $\varphi_n > \rho$ if $n > \mu$

Such an irrational number is called a positive irrational number.

We define $(\varphi_n) < \alpha$ if there is a negative rational number $-\rho$ such that $\varphi_n - \alpha < -\rho$ for all values of $n > \mu$, μ a positive integer.

$\therefore (\varphi_n) < 0$ if there exists a negative rational number $-\rho$ such that $\varphi_n < -\rho$ if $n > \mu > 0$; such an irrational number is called a Negative Irrational Number.

$(\varphi_n) > (\psi_n)$ if there exists a positive rational number ρ such that $\varphi_n - \psi_n > \rho$ for all values of $n > \mu$

$(\varphi_n) < (\psi_n)$ if there exists a negative rational $-\rho$

number such that $\varphi_n - \psi_n < -\rho$ for all values of $n > \mu$.

We will call the absolute value of (φ_n) $|(\varphi_n)|$ $\therefore |(\varphi_n)|$ $=(\varphi_n)$ or $(-\varphi_n)$ We will now prove the six fundamental theorems concerning the inequality of the irrational numbers.

Theorem I.

If $(\varphi_n) > (\psi_n)$ $(\psi_n) < (\varphi_n)$.

Proof; If $(\varphi_n) > (\psi_n)$,

$\varphi_n - \psi_n > \rho$ if $n > \mu$

Then $\psi_n - \varphi_n < -\rho$ if $n > \mu$

and $\therefore$ $(\psi_n) < (\varphi_n)$

can prove similarly if $(\varphi_n) \lessgtr \alpha$

$$\alpha \gtrless (\varphi_n)$$

Theorem II.

Of two unequal irrational numbers, one is greater and the other less.

Let (φ_n) and (ψ_n) be the unequal irrational numbers.

If $|\varphi_n - \psi_n| < \varepsilon$ for all values of $n > \mu$, $(\varphi_n) = (\psi_n)$ by the definition of equality of irrational numbers.

But by hypothesis $(\varphi_n) \neq (\psi_n)$

$\therefore |\varphi_n - \psi_n| \nless$ whenever $n > \mu$

$\therefore |\varphi_n - \psi_n| \geqq \varepsilon$ if $n > \mu$

Now choose $\rho < \varepsilon$. Then either $\varphi_n - \psi_n > \rho$ for all

values of $n > \mu$ or $\varphi_n - \psi_n > \rho$ for all values of $n > \mu$ and the theorem is proved by the definition of inequality of irrational numbers.

Theorem III.

$$\text{If} \quad (\varphi_n) > (\psi_n) \quad \text{and} \quad (\psi_n) = (\theta_n)$$
$$(\varphi_n) > (\theta_n)$$

Proof $\qquad \varphi_n - \psi_n > \rho \quad \text{if} \quad n > \mu$

$$|\psi_n - \theta_n| < \varepsilon \quad \text{if} \quad n > \mu'$$

Suppose $\qquad |\psi_n - \theta_n| = \varepsilon' < \varepsilon \quad \text{if} \quad n > \mu'$

$$\therefore \psi_n = \theta_n \pm \varepsilon' \quad \text{if} \quad n > \mu'$$

As ε is arbitrary we can make ε and $\therefore \varepsilon' < \rho$

(a) Now $\qquad \psi_n = \theta_n \pm \varepsilon' \quad \text{if} \quad n > \mu' \quad$ and

(b) $\qquad \varphi_n - \psi_n > \rho \quad \text{if} \quad n > \mu$

(a) & (b) both hold if $n > \bar{\mu}$, $\bar{\mu}$ being whichever is the greater μ or μ', or substituting in (b) for ψ_n its value from (a)

$$\varphi_n - \theta_n > \rho \pm \varepsilon' \text{ if } n > \bar{\mu}$$

and $(\varphi_n) > (\theta_n)$ by definition.

Theorem IV.

$$\text{If} \quad (\varphi_n) > (\psi_n) \text{ and } (\psi_n) > (\theta_n)'$$
$$(\varphi_n) > (\theta_n)$$

This theorem and theorem V can be proved in a manner exactly similar to that used in theorem III.

Theorem V.

If $(\varphi_n) > (\varphi'_n)$

and $(\psi_n) > (\psi'_n)$

$(\varphi_n)+(\psi_n) \not< (\varphi'_n)+(\psi_n)$ or $(\varphi_n)+(\psi'_n)$

Theorem VI.

Between any two unequal irrational numbers there must exist an irrational number different from either of them.

Suppose $(\varphi_n) > (\psi_n)$

Then $\varphi_n - \psi_n > \rho$ if $n > \mu$

or $\varphi_n > \psi_n + \rho$ if $n > \mu$

and $(\varphi_n) > (\psi_n) + \rho > (\psi_n)$

In a similar manner we can show, that between 0 and any irrational number there exists an irrational number. i.e. there is no smallest irrational number Similarly there is no largest irrational number.

Corollary;

There is no smallest or largest rational number

Addition of Irrational Numbers - - - - - -

We will define $(\varphi_n)+\alpha$ or $\alpha+(\varphi_n)$ where (φ_n) is irrational and α rational, as the irrational number $(\varphi_n+\alpha)$

We will define $(\varphi_n)+(\psi_n)$ or $(\psi_n)+(\varphi_n)$, where both (φ_n) and (ψ_n) are irrational, as the irrational number $(\varphi_n+\psi_n)$. From this definition and the laws for the addition of rational numbers we get when α & β are real numbers, rational or irrational,

1. $\alpha+\beta=\beta+\alpha$
2. $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$
3. If $\alpha=\alpha'$ $\quad \alpha+\beta=\alpha'+\beta$
4. If $\alpha>\alpha'$ $\quad \alpha+\beta>\alpha'+\beta$
5. If $\beta>0$ $\quad \alpha+\beta>\alpha$

That is, the laws for the addition of real numbers are the same as the laws for the addition of rational numbers. Hence the laws for the subtraction of real numbers are the same as the laws for the subtraction of rational numbers.

Multiplication of Irrational Numbers

We will define $\alpha\cdot(\varphi_n)$ or $(\varphi_n)\alpha$ where α is rational and different from zero and (φ_n) is irrational, as the irrational number $(\alpha\varphi_n)$ We will define $0\cdot(\varphi_n)$ or $(\varphi_n)\cdot 0$ as 0

We will define $(\varphi_n)\cdot(\psi_n)$ or $(\psi_n)\cdot(\varphi_n)$, where both are irrational, as the irrational number $(\varphi_n \cdot \psi_n)$ From this definition and from the laws for the multiplication of rational numbers we get, if a and b are two real numbers,

(1) $ab = ba$

(2) $(ab)c = a(bc)$

(3) $(b+c)a = ba + ca$

(4) If $a = a'$ $\quad ab = a'b$

(5) If $a > a'$ and $b > 0$ $\quad ab > a'b$

That is the laws for the multiplication of real numbers are the same as the laws for the multiplication of rational numbers. Hence the fundamental laws for the division of real numbers are the same as the fundamental laws for the division of rational numbers

In order to show that the new system is complete we now have to show that the remaining operation of limits leads us to a number of the system.

Theorem;

A regular sequence of φ's always has a limit in the new number system.

Hypothesis - $|\varphi_{n+r} - \varphi_n| < \varepsilon$ whenever $n > \mu$, $r = 1, 2, 3, \ldots$

To prove; there exists a number A of the system such that having chosen (1) $\varepsilon > 0$ we can determine (2) $\mu > 0$ such that

$$|A - \varphi_n| < \varepsilon \quad \text{whenever } n > \mu$$

First; Let φ_n be a sequence of rational numbers.

$$|\varphi_{n+r} - \varphi_n| < \varepsilon \quad \text{for all values of } n > \nu \quad r = 1, 2, 3, \ldots$$

$$\therefore \varphi_n - \varepsilon < \varphi_{n+r} < \varphi_n + \varepsilon \quad \text{whenever } n > \nu \quad r = 1, 2, 3, \ldots$$

Choose for n a fixed value, $n > \nu$ call it m.

$$\varphi_m - \varepsilon < \varphi_{m+r} < \varphi_m + \varepsilon$$

Let r be a variable.

φ_{m+r} is a regular sequence of rational numbers and by definition of irrational numbers it has a limit (φ_{m+r})

Call (φ_{m+r}), A and we will now prove A is the limit of φ_n.

$$\varphi_{m+r} - \varepsilon < A < \varphi_{m+r} + \varepsilon \quad \text{whenever } m > \nu$$

$$\therefore |A - \varphi_{m+r}| < \varepsilon \quad (r = 1, 2, 3 \ldots) \quad \text{Let } (m+r) = n$$

$$\therefore |A - \varphi_n| < \varepsilon \quad \text{whenever } n > \nu$$

hence A is the limit of φ_n.

$\therefore$ If φ_n is a regular sequence of rational numbers it has either a rational limit or by definition of irrational numbers an irrational limit.

We must also prove it has only one limit.

Suppose it had another limit $A + B = A'$

Then by definition of a limit we can choose (1) $\varepsilon > 0$ and find (2) $\nu' > 0$ such that $|A - \varphi_n| < \varepsilon$ whenever $n > \nu'$

But if it has another limit choosing the same $\varepsilon > 0$ we can find $\nu'' > 0$ such that

$$|A' - \varphi_n| < \varepsilon \quad \text{whenever } n > \nu''$$

Now calling the larger of ν' and ν'', $\bar{\nu}$ we have $|A - \varphi_n| < \varepsilon$

and $|A' - \varphi_n| < \varepsilon$ simultaneously for all values of $n > \bar{\nu}$

Now $|(A - \varphi_n) - (A' - \varphi_n)| \leqq |A - \varphi_n| + |A' - \varphi_n|$

since the absolute value of a difference is less than, or at most equal to the sum of the absolute values of its terms.

$$\therefore |A - A'| \leqq 2\varepsilon$$

$$\therefore |B| \leqq 2\varepsilon$$

and since there is no least number, rational or irrational, and the regular sequence of rational φ's has one and only one limit.

But the proof is not complete for, as we have enlarged our number system, it is possible to imagine a regular

sequence of irrational numbers, and, therefore, we must show also that a regular sequence of irrational numbers has a limit in the system.

In order to prove this we need the following;

Lemma;

There always exists a rational number differing from any irrational number by an irrational number ~~by an irrational number~~ less than any positive quantity we may choose.

Let (φ_n) be the irrational number, call it A

Then by definition,

$$|\varphi_{n+r} - \varphi_n| < \varepsilon \quad \text{whenever } n > \mu, \; r = 1, 2, 3, \dots$$

or

$$\varphi_n - \varepsilon < \varphi_{n+r} < \varphi_n + \varepsilon \quad n > \mu, \; r = 1, 2, 3, \dots$$

But by first part of the theorem φ_{n+r} has the limit $(\varphi_n) = A$

$$\therefore \varphi_n - \varepsilon < A < \varphi_n + \varepsilon \quad \text{if } n > \mu$$

or $|A - \varphi_n| < \varepsilon$ whenever $n > \mu$

and we can find a rational number φ_n differing from A by an irrational quantity less than any positive quantity we may choose.

Now let the given regular sequence of irrational numbers be $f_1, f_2, f_3, f_4,$ etc.

Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \varepsilon_4 > \varepsilon_5$ etc. be a set of ε's

chosen at pleasure

Form an arbitrary sequence g_1, g_2, g_3, g_4 etc.

such that $|g_n| < \varepsilon_n$ and $f_n - g_n = \varphi_n$

where φ_n is a rational number. This last condition is an allowable one by the preceding lemma

Will now show that φ_n is a regular sequence i.e. having chosen (1) $\delta > 0$ we can find a $\mu > 0$

such that $|\varphi_{n+r} - \varphi_n| < \delta$ whenever $n > \mu$, $r = 1, 2, 3, ----$

By hypothesis $|f_{n+r} - f_n| < \varepsilon$ $\quad n > \mu$, $r = 1, 2, 3, ---$

$$\therefore |f_{n+r} - f_n| = |\varphi_{n+r} + g_{n+r} - \varphi_n - g_n| < \varepsilon \quad \begin{matrix} n > \mu \\ r = 1, 2, 3 --- \end{matrix}$$

$$\text{or} \quad \varepsilon > |(\varphi_{n+r} - \varphi_n) + (g_{n+r} - g_n)|$$

$$\geqq |\varphi_{n+r} - \varphi_n| - |g_{n+r} - g_n|$$

since the absolute value of a sum is greater than or at least equal to the difference of the absolute values of its terms.

$$\therefore |\varphi_{n+r} - \varphi_n| \leqq |g_{n+r} - g_n| + \varepsilon$$

$$\leqq |g_{n+r}| + |g_n| + \varepsilon \qquad \left.\right\} \begin{matrix} n > \mu \\ r = 1, 2, 3, \cdots \end{matrix}$$

$$(K) \qquad < \varepsilon_{n+r} + \varepsilon_n + \varepsilon$$

Now choose arbitrarily a $\delta > 0$

Let $\varepsilon_n < \frac{\delta}{2}$ for all values of $n > \nu'$

$\varepsilon_{n+r} < \frac{\delta}{2}$ " " " " $n > \nu'$

Then $\varepsilon_n + \varepsilon_{n+r} < \delta$ for $n > \bar{\nu}$

$\bar{\nu}$ being whichever is the greater ν' or ν''

Let $\delta = \varepsilon_{n+r} + \varepsilon_n + \varepsilon$

This equation fixes the value of ε and by (K)

there exists a $\mu > 0$ such that

$$|\varphi_{n+r} - \varphi_n| < \delta \quad \text{if } \begin{matrix} n > \mu \\ r = 1, 2, 3, ---- \end{matrix}$$

$\therefore \varphi_n$ is a regular sequence and has the limit $(\varphi_n) \equiv A$

Will now show that A is the limit of f_n or having chosen (1) $\delta > 0$ we can find (2) $\mu > 0$ such that

$$|A - f_n| < \delta \qquad n > \mu$$

$$|A - \varphi_n| < \varepsilon \qquad n > \lambda$$

$$|A - f_n + g_n| < \varepsilon \qquad n > \lambda$$

$$|A - f_n| - |g_n| \leqq |A - f_n + g_n| < \varepsilon \qquad n > \lambda$$

$$|A - f_n| < \varepsilon + |g_n| \qquad \text{if } n > \lambda$$

$$< \varepsilon + \varepsilon_n < \delta \qquad \text{if } n > \lambda$$

$\therefore A$ is the limit of f_n.

Q. E. D.

and the irrational number system is complete.